Chapter 17

Logic Dynamics for Deductive Inference Its Stability and Neural Basis

Ichiro Tsuda

Research Institute for Electronic Science,
Hokkaido University, Sapporo, Hokkaido, Japan
tsuda@math.sci.hokudai.ac.jp

We propose a dynamical model that represents a process of deductive inference. We discuss the stability of logic dynamics and a neural basis for the dynamics. We propose a new concept of descriptive stability, thereby enabling a structure of stable descriptions of mathematical models concerning dynamic phenomena to be clarified. The present theory is based on the wider and deeper thoughts of John S. Nicolis. In particular, it is based on our joint paper on the chaos theory of human short-term memories with a magic number of seven plus or minus two.

1. Introduction

I first met John S. Nicolis in May 1983 when Hermann Haken organized the Synergetics meeting on the brain at Schloss Elmau in Germany.[1] John gave a talk entitled "The role of chaos in reliable information processing", which was very impressive.[2] Surprisingly, John knew of my several papers on the mathematical modeling of chaos and bifurcations in the Belousov-Zhabotinsky reaction, coauthored with the late Professor Kazuhisa Tomita, and of the paper on noise-induced order, coauthored with my younger colleague in the Tomita laboratory, Kenji Matsumoto. John was very enthusiastic about discussing on these matters with me, and about explaining his own ideas on chaotic information processing.[1,3] His ideas on this subject were fascinating and immediately attractive to an adolescent and ambitious mind.

After returning to Japan, and being influenced by his deep and generous thoughts, I developed an idea how to calculate information storage capacity in chaotic dynamical systems. I calculated its values for Rössler and Lorenz attractors and sent these results to John by airmail. About 20 days later, I received a return airmail containing a draft for a joint paper. We then exchanged several airmails to confirm our mutual agreement about fundamental ideas, calculation results, and the organization of the paper. Finally, we submitted the paper to the Bulletin of Mathematical Biology, which became our first joint paper.[4] In the paper, we treated the magic number "seven plus or minus two" which was recognized as the capacity of human short-term memory in terms of both the Lyapunov spectrum and the fluctuations of local divergence rates in chaotic dynamical systems.

Concerning the information structure of chaos, Oono[5] first studied Kolmogorov-Sinai entropy in chaotic dynamical systems, and Shaw[6] proposed the concept of information flow in chaotic dynamical systems. Stimulated by the studies of Oono, Shaw, and John Nicolis, Matsumoto and I also studied the information structure of chaotic behavior, for which we proposed the concept of the fluctuations of information flow, and a method of calculation for such fluctuations in terms of conditional mutual information in a bit space.[7–9] We also applied these information-related theories to the information processing in the brain, via the framework of hermeneutics of the brain.[10,11]

With respect to the mathematical modeling of the brain and mind in the field of cognitive neuroscience, various levels of description from the single neuron level to the level of a society of brains have been proposed so far. John Nicolis' studies covered all levels of description. He also addressed essential but hard problems such as bridging between neural activity and cognition.[1,12,14–16] The nonlinear dynamics of games that John Nicolis treated, can be classified as a study at the level of cognitive neurodynamics. Later, it turned out that this approach, in addition to our own approach,[4] is similar to that of Grim and Mar,[17–19] which describes the inference process with fuzzy logic in terms of discrete-time dynamical systems.

My own interests have lain in the dynamic relationship between memory and thoughts.[20] It is well known that episodic memory is stored in the temporal cortex after the episodic signals pass through the hippocampus, which is responsible for the transformation from short-term to long-term memory. Working memory operates over a few seconds, in order to manipulate information, to make a temporary storage, and to focus attention via interactions among the prefrontal cortex, cingulate cortex, parietal cortex, and basal ganglia. Therefore, working memory includes the short-term memory related to inference processes, such as the depth of recursive inference. Our joint paper on the magic number "seven plus or minus two" was about a chaotic theory for working memory in this sense. Furthermore, now it turns out that the prefrontal cortex, particularly, the dorsal lateral prefrontal cortex, is responsible for inference based on conditional associations.[21] On the other hand, deliberative decision-making has been observed in human and some animal behavior during a learning process.[22] Human beings and even animals necessarily deliberate at a decision point in space and time to make a true judgment. This process, from deliberation to final judgment, must involve the internal dynamic processing of truth values for the hypothesis posed, based on past experience, that is, based on memories.

In digital computer systems, "inference processes" can be performed in terms of a computation unit and a bit space where both computational results and external data are memorized, with computation and memory operating separately. In other words, the memory system and the inference system can be separated in digital computers. However, in human and animal brains, it seems that these two systems do not operate separately. The two systems interact with each other, particularly those interactions between the short-term memory of events and the sequence of inferences on those events that typically result in episodic memory. In this respect, it is hypothesized that episodic memory is a representation of a prototype of inference.

In relation to this hypothesis, we have proposed a dynamic theory for episodic memory, the Cantor coding theory. In this theory, dynamic transitions of neural activity states such as chaotic itinerancy in CA3 of the hippocampus play a role in reconstructing a series of episodes, and contraction dynamics in CA1 of the hippocampus can form Cantor sets in the state space of neural activity, each element of which represents an episode.[20,23–25] This theory has been proven in a rat slice experiment,[26,27] and it is anticipated that it will include changes via synaptic learning, such as Tsukada's learning rule.[28–30] Although the theory has not yet been proved in human and intact animal brains when undergoing episodic experiences, it suggests a similar coding scheme, using chaos and fractal geometry for the neural representation of human and animal inference. Here one can see John Nicolis' fundamental ideas on the interplay between chaos and fractal.[13]

Historically, research on inference has developed in association with research on thought processes, going back to, for instance, Aristotle, Hobbs and Leibniz. However, George Boole's ideas[31] introduced a radical new approach. He considered the laws of thought, derived the binary values, 0 and 1, and tried to clarify the relationship between logic and probability in terms of mathematics. His thoughts influenced the research of Turing, McCulloch and Pitts, and von Neumann on the realization of human thought by means of computation in digital computer or neural networks.

The present paper treats typical deductive inference processes in relation to dynamical systems. It can be considered as an essay on the dynamics of thought. We start with the origins of Boolean logic and try to extend Boolean logic to the area of cognitive neurodynamics, or mental movement, introducing a discrete time step to represent the neural delays stemming from both the absolute refractoriness of neurons and the delayed feedback in neural networks. The discrete-time dynamical systems introduced in this way are similar to those treated by Grim and Mar.[17–19] We describe this issue with inference processes about typical ambiguous statements in Section 2. In Section 3, we further treat continuous-time dynamical systems as a limit of infinitesimal time lapses in discrete-time dynamical systems. In Section 4, a neural basis for finite time is treated. In Sections 5 and 6, we treat description dynamics and its stability, respectively. Section 7 is devoted to summary and discussion.

2. Logical Inference and "Step Inference"

We start with a brief review of the origin of binary logic; that is, classical logic. George Boole invented binary logic and published a book[31] entitled "An investigation of the laws of thought" in 1854, in which he queried the origin of thought. He identified thought as determining the truth or falsehood of given statements, and he tried to construct a mathematical basis for logic and probabilities, thereby trying to make clear the laws of human intellect. For the first time, he tried to deduce the binary values 0 and 1, using the following procedure. He first asked whether, for example, "Blue Blue" is "Blue". If so, $xx = x$, where "Blue", and the symbol "=" denotes the identity of classes. In this symbolic expression, he represented the identity of the class of blueness. Because human inference is based on certainty in the identification of object classes, he introduced the product operation in juxtaposition of, regarding a variable as a certainty. He then obtained the algebraic equation, $x^2 = x$ or $x(1 - x) = 0$. The solutions of this equation are simply 0 and 1. These binary values can be considered the truth values of the statement that "Blue Blue" is "Blue". For him, "1" and "0" implied "God" and "the others", respectively. He therefore considered that a reconstruction of the world in terms of these binary values is possible, where the world is typically represented by mathematics.

Here we extend the Boole's method by the explicit introduction of a unit of time as a unit in the process of inference. To do this, we introduce a dynamical system associated with the inference process that determines the truth values of statements, as in both Grim's framework[17,19] and our framework.[32] In *logical inference*, obtaining consequence from premise is usually assumed to be instantaneously performed, but it will take a certain time in the *human inference process*. Furthermore, we ordinarily use a *recursion process* to determine the truth value of a given statement. In other words, we repeat a combined process of two subprocesses: deduction from premise to consequence according to logic, and substitution of the consequence with the premise for the next step of inference. Let the premise be P, and let the consequence be C. There are two main ways to introduce a time step n: in the process from premise to consequence, and in the process of substitution of consequence with premise. For the former case, we obtain

$$X_{n+1}(C) = F(X_n(P)) \tag{1a}$$
$$X_{n+1}(P) = X_{n+1}(C) \tag{1b}$$

whereas for the latter, we obtain

$$X_n(C) = F(X_n(P)) \tag{2a}$$
$$X_{n+1}(P) = X_n(C) \tag{2b}$$

where X denotes the truth value of the statement, and F denotes the transformation of the truth value for the deductive inference.

For either case, we obtain

$$X_{n+1}(C) = F(X_n(C)) \tag{3}$$

In some special cases, this reduction in Eq. (3) does not lead to a correct decision, because the two processes given by Eqs. (1) and (2) lead to different truth values[32] (see also Section 4). However, in the present paper, we consider the reduction by Eq. (3) as giving a correct decision. Let us call this type of inference a *step inference*.

Now, consider a dynamical system of inference for Boole's blue. It is straightforward to obtain a corresponding map.

(1) *The statement of Boole's blue.*

$$X_{n+1} = X_n^2 \tag{4}$$

Here, X is a real number in $[0, 1]$, representing a truth value. The binary values 0 and 1 that Boole derived are obtained as fixed points in this dynamical system. However, the asymptotic solution is $X = 0$, which is an attractor. In the following, we will treat dynamical systems corresponding to slightly more complex statements, which typically seem to show the processes of inference in human mind, as well as the inference process of Boole's blue. Here, we use similar statements to those that Grim used,[19] where he adopted fuzzy logic and obtained chaotic behavior associated with a step inference.

(2) *This sentence is false.* Let this statement be denoted by X. The statement can then be replaced by X is false. Hereafter, we use the same symbol for the truth value as for the statement. The discrete-time dynamical system, that is, the map, which represents the inference process of determining its truth value, is given by the equation

$$X_{n+1} = 1 - X_n \tag{5}$$

The fixed point is $X = 1/2$, which cannot be achieved in classical logic because of the law of the excluded middle. Of course, if one extends the logic to multivalued logic, $X = 1/2$ is acceptable as an "I don't know" state. Restricted to classical logic, this equation of motion, Eq. (5), has an oscillatory solution; that is, a period-two solution, $\{X_n = 0, X_{n+1} = 1\}$. In classical logic, therefore, this statement is *undecidable*. If one extends the logic to multivalued logic, the truth values satisfying the equation of motion are infinitely many, that is, $\{X_n = s, X_{n+1} = 1 - s, (s \in [0, \frac{1}{2}])\}$, all of which are period-two solutions. The result of a step inference is equivalent to that of logical inference.

(3) *This sentence is true.* Let this statement be denoted by X. The statement can then be replaced by "X is true" Similarly, the discrete-time dynamical system is given by the equation,

$$X_{n+1} = X_n \tag{6}$$

In classical logic, the solutions are given by the fixed points of the dynamical system, $X = 0$ and $X = 1$. This statement is therefore *indeterminate*. Extending to multivalued logic, all numbers from 0 to 1 represent solutions. The result of a step inference is equivalent to the one in logical inference.

(4) *The sentence X: the next sentence Y is false. The sentence Y: the previous sentence X is false.* The equations of motion determining these truth values are as follows:

$$X_{n+1} = 1 - Y_n \tag{7a}$$

$$Y_{n+1} = 1 - X_n \tag{7b}$$

The fixed points associated with classical logic are $(X, Y) = (1, 0)$ and $(X, Y) = (0, 1)$ Extending to multivalued logic, all numbers $X, Y = 1 - X \in [0, 1]$ represent the solutions of Eq. (7). The consequence is that a step inference is equivalent to a logical inference, both of which lead to indeterminacy. However, one can find a new solution, that is easily obtained by a step inference. This other solution of Eq. (7) is *oscillatory*, such that $\{(X_n, Y_n) = (0, 0), (X_{n+1}, Y_{n+1}) = (1, 1))\}$. This solution has been excluded in the conventional consequences of logical inference. Because this solution represents undecidability in the statement, the consequence allows a higher level of contradiction, in that the statement implies both undecidability *and* indeterminacy. Two sentences X and Y are contradictive in the sense of conventional logical inference, because neither $X \cap Y$ nor $\neg X \cap \neg Y$ hold, where denotes $\neg$ negation. However, under a step inference, these two sentences are *not* contradictive, because both because both $X \cap Y$ and $\neg X \cap \neg Y$ hold at different time steps, because of the presence of a period-two solution.

Because the consequences for the truth value of a pair of these sentences are different for logical and step inference it is worth studying the cause of this difference. We will treat this issue in the next section.

3. Introduction of Infinitesimal Time: "Differential Inference"

Let us assume that Eq. (7) was derived by Euler's method applied to certain differential equations. Using this assumption, we will find the differential equations corresponding to the inference process of the truth value of the pair of sentences mentioned in the previous section. From Eq. (7), $X_{n+1} - X_n = 1 - X_n - Y_n$ and $Y_{n+1} - Y_n = 1 - X_n - Y_n$ obviously follow. If a unit time, that is, a time step 1 is viewed as a time step corresponding to an infinitesimal time scale, then we can find the following differential equations.

$$\frac{dX}{dt} = 1 - (X + Y) \tag{8a}$$

$$\frac{dY}{dt} = 1 - (X + Y) \tag{8b}$$

This is, of course, a first-order approximation to the difference equations in Eq. (7), in terms of differential equations. In fact, the set of differential equations equivalent to the set of difference equations given by Eq. (7) is the first order of the infinitely many simultaneous differential equations that include those having the same terms in the right hand side of the equations as those in Eq. (7). This relationship between the two expressions, in terms of infinite-dimensional differential equations and finite-dimensional difference equations, may stem from the following features of the shift operator $e^{\frac{\partial}{\partial n}}$, where n is supposed to be extended to the real[33,34]:

$$Z_{n+1} = e^{\frac{\partial}{\partial n}} Z_n = \left(1 + \frac{\partial}{\partial n} + \frac{1}{2!}\frac{\partial^2}{\partial n^2} + \cdots \frac{1}{k!}\frac{\partial^k}{\partial n^k} + \cdots \right) Z_n \tag{9}$$

Applying the expression (9), the original difference equation, $Z_{n+1} - Z_n = f(Z_n)$, can be transformed via infinite-dimensional differential equations in the following way.

Set $Z_n^{(1)} = \frac{\partial}{\partial n} Z_n$, which provides the first equation. The second equation is obtained by setting $Z_n^{(2)} = \frac{\partial}{\partial n} Z_n^{(1)}$. Similarly, for the kth equation, $Z_n^{(k)} = \frac{\partial}{\partial n} Z_n^{(k-1)}$. Finally, $\frac{\partial}{\partial n}\left(Z_n + \frac{1}{2!}Z_n^{(1)} + \cdots + \frac{1}{(k+1)!}Z_n^{(k)}\right) = f(Z_n)$, $(k \to \infty)$. Because each order of derivative becomes a base for a $j + 1$ dimensional vector space that comprises linear combinations of derivatives up to the jth order, all the variables $Z_n^{(i)}$ (supposing $Z_n = Z_n^{(0)}$) except for the final variable are independent of each other.

Here, we use a first-order approximation of this formula as the above differential approximation, such as

$$\frac{dZ}{dt} = f(Z) \tag{10}$$

using the same symbol t as in Eq. (8) in place of n, and replacing ∂ with d for the derivative. The second approximation will be

$$\frac{dZ}{dt} = Z^{(1)}, \tag{11a}$$

$$\frac{1}{2}\frac{dZ^{(1)}}{dt} = -Z^{(1)} + f(Z). \tag{11b}$$

The third approximation will be

$$\frac{dZ}{dt} = Z^{(1)}, \tag{12a}$$

$$\frac{dZ^{(1)}}{dt} = Z^{(2)}, \tag{12b}$$

$$\frac{1}{3!}\frac{dZ^{(1)}}{dt} = -Z^{(1)} - \frac{1}{2}Z^{(2)} + f(Z) \tag{12c}$$

and so on.

It is clear that the fixed points in any order of differential approximations are the same as in the original difference equations. The stability of these fixed points is, however, nontrivial when they change and how they change, even considering the fact that they change within the limit of the approximation.

The asymptotic solution of Eq. (8) is $X + Y = 1$, and the period-two solution disappears. In classical logic, this means that $(X, Y) = (1, 0)$ or $(0, 1)$; that is, it is an indeterminate statement. In other words, the consequence of conventional logical inference is recovered by eliminating an undecidable solution. Let us call this type of dynamical inference by differential equations a *differential inference*.

The consequence by differential inference implies that Sentence 4 in the previous section includes contradiction in a sense of logical inference, which can be described by continuous-time dynamical systems defined with infinitesimal time, but escapes this type of contradiction in a step inference, which can be described by a discrete-time dynamical system that introduces a finite width of time such as a time step. This method of overcoming the difficulty, namely contradiction, is a consequence of the appearance of a period-two solution in the inference process.

Dynamical systems in continuous time corresponding to the other sentences yield the solutions for truth values obtained by logical inference.

For Sentence 1, the asymptotic solution in differential inference is $X = 1$. For Sentence 2, it is $X = 1/2$, which implies ambivalence or no solution in classical logic. For Sentence 3, it is $X = const.$, which implies indeterminacy.

Variables treated here are truth values of statements, which imply certainties of decision-making via deductive inference, and thus time-varying certainties were studied. Correspondingly, decision-making in self-referential paradoxical games was studied by Nicolis *et al.*,[35] where time-varying probabilities of cooperation were described by differential equations, and those equations possessed fixed points as the solutions representing contradictory states.

4. The Neural Basis of Finite Unit Time

As shown in the previous sections, the introduction of a finite unit of time in inference processes yields an oscillatory solution for truth values, thereby avoiding contradiction. One of our assertions here is that human beings adopt step inferences in their decision making in daily life. This idea can be applied to experiments on animal behaviors based on an inference process,[36] such as transitive inference[21]: if $A \to B$ and $B \to C$, then $A \to C$, where the arrow ($\to$) denotes implication. An animal's ability of transitive inference may be a basis for human deductive inference or syllogism. It may also be a basis for decision making, even in circumstances involving inconsistent events, where inference and decision making must be performed via step inference.

A question arises: what is the origin of the unit of time in step inference? The most basic time step in neural systems is the absolute refractory period of a single neuron. At the network level, delayed-feedback connections can yield a unit of time. Consider the following two typical cases: Case (a), where the absolutely refractory period is rate-determining, and Case (b), where the feedback delay time is rate-determining.

We further introduce relative refractoriness, as in Aihara's neuron model.[37] One merit of using this model is that the model not only includes an absolute refractory period as a unit time step but also includes relative refractoriness in the form of an exponential decay of memory, which produces differences in the effects of delayed feedback. We can then obtain the following equations of motion for the neural activity of a recurrent neural network.

Case (a):

$$x_{n+1}^i = \sum_j w_{ij} y_n^j \tag{13}$$

$$y_{n+1}^i = f(x_n^i - \sum_{k=0}^{n} b^k y_{n-k}^i - \theta^i) \tag{14}$$

where f denotes a transformation function f from input to output, w_{ij} is the coupling strength from the jth neuron to the ith neuron, b $(0 < b < 1)$ is the decay rate of memory, and θ^i is the threshold for the ith neuron.

Let X_n^i be the effective membrane potential of the ith neuron at time n. The overall equation rewritten in terms of is then as follows:

$$\begin{aligned} X_{n+1}^i = bX_n^i - f(X_n^i) + \sum_n^i w_{ij} f(X_n^i) \\ - b \sum_j^n w_{ij} f(X_{n-1}^j) - (1-b)\theta^i \end{aligned} \tag{15}$$

This results in a chaotic neural network.[37]

Case (b):

$$x_{n+1}^i = \sum_j w_{ij} y_n^j \tag{16}$$

$$y_{n+1}^i = f(x_{n+1}^i - \sum_{k=0}^{n} b^k y_{n+1-k}^i - \theta^i) \tag{17}$$

Let X_{n+1}^i be the effective membrane potential of the ith neuron at time $n+1$. The overall equation rewritten in terms of is then as follows:

$$\begin{aligned} X_{n+1}^i = bX_n^i - f(X_{n+1}^i) + \sum_j w_{ij} f(X_n^j) \\ - b \sum_j w_{ij} f(X_{n-1}^j) - (1-b)\theta^i \end{aligned} \tag{18}$$

This is a bootstrap type of equation of motion. In other words, one should solve the functional equation, $X + f(X) =$ a previously calculated value, at each time step. This may also result in another chaotic neural network. In fact, if $f(X)$ is a sigmoid function and its derivative at the origin is greater than 1, then Case (b) will yield much more stable activity of neurons than Case (a). Otherwise, it may yield unstable dynamics, giving rise to chaotic behavior in the overall network, as for Case (a). However, with respect to the appearance of future time on the right-hand side of the equation, it is still questionable whether this future time would bring about essentially new features in the dynamic behavior, different from the formal differences between Ito and Stratonovich integrals in stochastic calculi.

5. Description Dynamics for External Phenomena

In the previous sections, we assert that human and even animal inference is performed in the form of step inference, and the origin of the unit of time in such inference lies in an absolute refractory period or in a delay time associated with feedback connections. Human beings and animals infer a truth value for an event after transforming that event in the form of descriptions; that is, sentences. So far, we have restricted ourselves to treating the process after such transformations. In this section, we treat the dynamics of description that may occur in the brain before and after the evaluation of the truth value for the event.

Let us assume that phenomena occurring in the external environment can be described by dynamical systems. In other words, we assume that even when deterministic systems are perturbed by external noise, the overall dynamics can be described by skew product transformations of the dynamical systems and small- amplitude chaotic systems producing a given stochastic process. Internal dynamics in the brain can be active in describing these external dynamics $X(t)$. Let us denote the dynamics associated with such a description by $h(X(t))$. There could be two extreme states for such a description: completely adaptive state such as $h(X(t)) = X(t)$ and an indifferent or "autistic" state such as $h(X(t)) = const$. The actual states of the internal description must be intermediate between these extremes.

To describe the dynamics of the intermediate states more explicitly, let us adopt discrete-time dynamical systems for both the internal and external dynamics. For the external dynamics, we adopt, $X_{n+1} = F(X_n)$ where X_n is an element in N-dimensional vector space, subscript n is a discrete time step, and F is a differentiable map. When we observe and describe this type of dynamical system, the dynamics of the internal description $h_{n+1}(F)$, which represents some neural activity in the brain, can be described by another map $\tilde{F}$. The *description dynamics* is therefore as follows:

$$h_{n+1}(F) = \tilde{F}(h_n) \tag{19}$$

More explicitly, representing the above formula in terms of external states:

$$h_{n+1}(X_{n+1}) = \tilde{F}(h_n(X_n)). \tag{20}$$

In this formula, the above two extreme states are formulated as follows.

(1) A completely adaptive state is formulated by obtaining an invariant h under the condition that $\tilde{F} = F$. A trivial solution is given by $h(X) = X$, which implies making a copy of the external world.
(2) An indifferent state is formulated under the condition that $\tilde{F} = X$, which provides the fixed points for the internal dynamics. Then, $h(X_{n+1}) = h_n(X_n)$, that is a fixed description, which implies an independent description of the external world.

The actual state provided by the description dynamics will be obtained as a solution for the following functional equation of motion:

$$h_{n+1}(F(X_n)) = (1 - \varepsilon)F(h_n(X_n)) + \varepsilon h_n(X_n). \tag{21}$$

where ε is a parameter representing a balance between the above two extreme states, which can be a bifurcation parameter. This equation covers the situation where the right-hand side of the equation represents $\tilde{F}$.

It should be noted that this functional equation of motion can represent useful systems, such as the Kataoka-Kaneko functional map,[38] which can be realized by the condition that $F(X_n) = X_n$ externally and $F = h$ internally. In such a case, we would obtain

$$h_{n+1}(X_n) = (1 - \varepsilon)h_n(h_n(X_n)) + \varepsilon h_n(X_n). \tag{22}$$

This functional map has been further investigated mathematically by Takahashi and Namiki, who proved the existence of a hierarchical structure of periodic solutions.[39–41]

In the Kataoka-Kaneko formula, the presence of the self-referential term of description in Eq. (22) is essential for representing the complexity of the dynamics, but it makes analysis difficult. This may imply the impossibility of neural activity dealing directly with self-referential descriptions. When neural systems process a self-referential description, they may first have to make a copy of the object of self-reference and then refer to this copy. This two-stage formulation can be realized mathematically in the proof of Gödel's incompleteness theorem through the processes of projecting mathematical statements to natural numbers and of referring to meta-mathematical statements by providing mathematical statements about such numbers. The presence of mirror neurons in animal brains[42] or mirror-neuron systems in human brains[43] may also be a realization of the above two-stage formulation in brains, because mirror neurons, or mirror-neuron systems, can be activated, not only by behavior in others similar to one's own behavior, but also by one's own behavior. This can be represented in a dynamical systems model.[44]

6. Descriptive Stability

Combining descriptive dynamics with the dynamics of truth value x, some functionof $G(x)$ of x implies a certainty of description with respect to the truth value. The dynamics of this certainty described by functional maps such as those mentioned in the previous section can therefore describe the dynamics of decision making. One of the important questions will be the stability of such a description. We have tried to formulate it in a similar way to the definition of the pseudo-orbit tracing property of dynamical systems.[32] The pseudo-orbit tracing property is defined following Robinson.[45]

Let h be a continuous map on a compact space M. For $x \in M$, $\{h^{(i)}(x)\}_i$ represents an orbit on M. The observed orbit is, however, not always identical to the dynamical orbit, because of round-off errors in computers or external noise or perturbations in laboratory experiments. Let $\{y_i\}_i$ be an observed orbit. If there exists $a > 0$ such that for any i, $h(y_{i-1})$ is in an α-neighborhood of y_i, then the observed orbit is called an α-pseudo-orbit. If for some $x \in M$ there exists $\beta > 0$ such that for any n, $h^{(n)}(x)$ is in a β-neighborhood of y_n, then the pseudo-orbit $\{y_i\}_i$is β-traced by x. If any α pseudo-orbit is β-traced, then the dynamical system (h, M) possesses a pseudo-orbit-tracing property. The pseudo-orbit-tracing property indicates the stability of dynamical systems associated with observations, which is related to structural stability.[46]

The stability of dynamical systems associated with descriptions can be defined in a similar way. Here, we use the same symbols as those used in the previous section. We have one finite-dimensional dynamical system (F, L) and another infinite-dimensional dynamical system on function space $(\tilde{F}, W)$, where $h \in W$. If there exists $\alpha > 0$ such that $F \circ h_{i-1}^{-1} \circ \tilde{F}^{i-1}$ is in an α-neighborhood of $\tilde{F}^i$ for any i, then we call $\tilde{F}$ an α pseudo-dynamical system. If for some description g in description space, which is assumed to be compact, there exists $\beta > 0$ such that for any n, $F^n \circ g^{-1}$ is in a β-neighborhood of $\tilde{F}^n$, then the pseudo-dynamical system $\tilde{F}$ is β-traced by g. If any α pseudo-dynamical system is β-traced, then the dynamical system (F, L) possesses a pseudo-dynamical system-tracing property. We would like to propose the concept of *descriptive stability*, using this pseudo-dynamical system-tracing property.

When we try to apply this new stability concept to the inference processes defined by step inference, we have to assume the external dynamics F, that is the subject for the inference dynamics in our mind. For example, F could describe some reaction of macromolecules for the activation of receptors. In such a case, we would describe the activity of receptors such that the receptors are active when a macromolecule A is attached. The truth value of this statement, for example, would depend on the probability of that attachment, which may change over time. We can obtain a description of truth values by a map h. Its dynamics could be discussed by, say, the introduction of logic dynamics, represented $\tilde{F}$ by. We now assume that the external dynamics is described by differential equations. We also assume that human minds will always use step inferences. The external dynamics that can then be formulated by Sentences 1–3 in Section 2 possesses descriptive stability, because for external dynamics F, the internal description via step inference $\tilde{F}$ provides the same result as F. However, the dynamics corresponding to Sentence 4 has an unstable description, because a step inference $\tilde{F}$ can provide a completely different result from F. On the other hand, if the external dynamics is described by difference equations, then the external dynamics corresponding to Sentences 1–4 do possess descriptive stability.

7. Summary and Discussion

Motivated by George Boole's way of thinking about Boolean logic and by John Nicolis' way of thinking about chaotic information processing, we obtained the dynamics associated with inference processes via the introduction of the concept of step inference, which is similar to Grim's theory. We first studied the relationship between logical inference and step inference, finding that differential inference as an infinitesimal time-step version of step inference can act as a dynamical model for logical inference. We also found typical examples for which step inference produced different consequences from logical inference. Within the framework of step inference, contradiction in logical inference can disappear by virtue of the appearance of a finite unit of time, which might be a basis for natural behavior in living systems. However, this finding does not preclude that game-theoretic inference described by differential equations can realize contradictory states. Indeed, J. S. Nicolis showed[35] that differential inference for decision-making in self-referential paradoxical games allows contradictory states such that both players could win or lose if state dependent probabilities of cooperation are introduced. Contradictory states are here realized as an alternative switching between two fixed points expressing that sentences are true or false, which typically corresponds to the solutions of Eq. (7).

We further provided a neural basis for this kind of finite unit of time. In particular, we treated two typical cases, formulated by different types of equations of motion. Contrary to the conventional viewpoint, delayed feedback may yield a super-stable steady motion, to the extent that discrete-time dynamics modeled by difference equations is adopted, which support step inference.

Furthermore, we formulated stability of description, introducing the new concept of descriptive stability. We provided concrete examples of descriptive stability in relation to the logical sentences posed as typical objects of inference.

Here, we treated only some examples of deductive inference. However, the theory can be extended to other complex processes of inference, such as procedures whereby applied mathematicians try to make mathematical models of natural phenomena. In modeling the dynamics of a certain phenomenon, they first try to create a clear description in terms of sentences for the dynamic process of that phenomenon. They then transform the description into equations that correctly represent the dynamic process that should be the essence of the phenomenon.[47] Therefore, it is necessary to consider the descriptive stability of the phenomenon concerned. In particular, it should be noted that choosing which differential and difference models should be adopted is often crucial because a sentence-based description of the dynamic process is consistent with step inference but is not always consistent with logical inference.

It should also be noted that chaotic behaviors, which can appear in both step inference and differential inference, could play an important role in decision-making. Chaotic dynamics of truth values constitute an invariant set concerning certainties of inference process. Therefore, in the convergent process of certainties, we observe deliberative decision-making during transient motion of the dynamics, and also convergent thinking with probabilities in an asymptotic stage of the dynamics. Thus chaos plays a role in providing flexibility of decision-making even if the system concerned includes contradiction, as clearly stated[3] by John Nicolis.

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